LU Decomposition and Cramer's Rule Linear Algebra X. Du

- LU Decomposition
  - Proposed by Alan Turing
  - The aim is to decompose a square matrix A into the form $A = LU$ (or $A = P^T LU$), where L is lower triangular, U is upper triangular, and P is a permutation matrix.
  - Steps:
    - Complete Gaussian elimination, without interchanging rows or multiplying rows, by using operations of the type $-l_{ij}R_j + R_i$ until you have an upper triangular matrix U.
      - Note that operations of the type $-l_{ij}R_j + R_i$ do not change the determinant, so $\det(U) = \det(A)$
    - Form a lower triangular matrix L using l_{ij} as entries below the diagonal and 1's on the diagonal.
    - If you need to exchange rows, then you will need a permutation matrix P that exchanges the rows, but be careful to keep track of variables
      - Example of P: switching the first two rows of a 3-by-3 matrix:
        $$P = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
    - Note that LU-factorizations are not unique.
      - For example, you could alternatively use division to reduce your pivots to 1's in the upper triangular matrix before performing the $-l_{ij}R_j + R_i$ operations. Then the diagonal of the upper triangular matrix should be 1's, and the diagonal of the lower triangular matrix should be factor you divided each individual row by.
  - Applications of LU decomposition: if you have $A\vec{x} = \vec{b}$ with the form $A = LU$, then you can substitute to get the equation $LU\vec{x} = \vec{b}$. Then you can easily solve two systems of equations $L\vec{y} = \vec{b}$ and $U\vec{x} = \vec{y}$ using back substitution to solve for $\vec{x}$
- LDU Decomposition
  - The aim is to decompose a square matrix A into the form $A = LDU$, where L is a lower triangular matrix and U is an upper triangular matrix, both with 1's on the diagonals, and D is a diagonal matrix.
  - The process is the same as previously outlined for LU decomposition, except that you factor out the rows of the upper triangular matrix so that you have 1's as the pivots, and place those factors onto the diagonal of D.
- **Cramer's Rule**
  - Given $A\vec{x} = \vec{b}$ and $\det(A) \neq 0$ (i.e. A invertible), the unique solution to $A\vec{x} = \vec{b}$ is $x_1 = \frac{\det(A_1)}{\det(A)}, x_2 = \frac{\det(A_2)}{\det(A)}, \ldots, x_n = \frac{\det(A_n)}{\det(A)}$, where A_i is the matrix of the ith column vector of A replaced by $\vec{b}$